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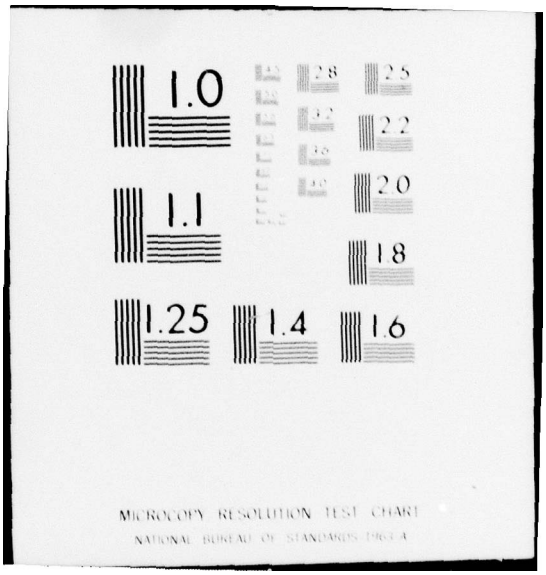
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by

Philip Heidelberger

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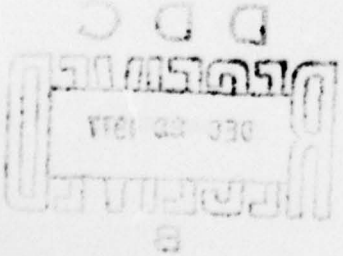

VARIANCE REDUCTION TECHNIQUES FOR SIMULATING MARKOV CHAINS

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ABSTRACT

Simulators frequently wish to estimate parameters of the limiting distribution of stable stochastic processes. Several new methods for reducing the variance of such estimates will be proposed and discussed. The methods are applicable to regenerative Markov processes in both discrete and continuous time as well as to semi-Markov processes. The methods are similar to the technique of multiple control variables yet differ in the important respect that it is not necessary to calculate the means of the controls. This is because the controls are chosen in such a way that their means actually equal the parameter of interest. The methods do require a certain amount of computation to be done before the simulation begins, although their cost should be relatively minor compared with that of the simulation. Numerical results demonstrating the effectiveness of the techniques for a simple queueing model are presented.

I. INTRODUCTION

In recent years computer simulation has become a very important tool for analyzing the behavior of stochastic processes. As the structures of widely used processes become increasingly complex analytic results become more difficult to obtain so that frequently simulation is the only computationally feasible method to study a process. Unfortunately simulation can be a very expensive tool to use. It is therefore desirable to develop methods that can reduce the run lengths (and hence cost) of a simulation yet still give accurate estimates. Such methods are called variance reduction techniques. This paper will propose several new variance reduction techniques in the special case when the stochastic process being simulated is a Markov process.

The methods are all control variable techniques (see [5]), yet they differ from most other control methods in the important respect that the means of the controls need not be explicitly known. This is because the controls are chosen in such a way that their means actually equal the parameter of interest. The methods require a certain amount of computation to be done both before and during the simulation but hopefully their cost will not be so great as to prohibit the use of the methods.

II. MARKOV CHAINS

We begin by introducing notation and stating some preliminary results for Markov chains. The reader should consult [1], [2], or [3] for a more detailed analysis of these stochastic processes.

Let $\{X_n, n \geq 0\}$ be a Markov chain with countable state space $E = \{0,1,2,\ldots\}$ and n-step transition matrix P^n. We assume the chain is irreducible, aperiodic and positive recurrent. Under these conditions the following Proposition is true.

(1) PROPOSITION. There exists a probability distribution π on E and a random variable X with distribution π such that

$$P_{ij}^{\,n} \to \pi_j > 0 \quad \text{for all } j \in E \tag{2}$$

$$X_n \Rightarrow X \tag{3}$$

$$\pi = \pi P, \text{ i.e. } \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} . \tag{4}$$

The " $\Rightarrow$ " in (3) denotes convergence in distribution. Now let f_j be a real valued function on

E and define

$$r_j = \pi f_j = \sum_{i=0}^{\infty} \pi_i f_j(i) = E[f_j(X)] \qquad (5)$$

It is frequently of interest to know r_j for a variety of functions f_j. If the state space is very large (perhaps infinite) the set of stationary equations (4) may be very difficult to solve so that r_j must be estimated via simulation. It is the efficient estimation of such quantities that we will concern ourselves with. We now develop an alternate expression for r_j which will be useful in simulation.

Pick some sate in E (called the return state) which will be designated by 0. Let $X_0 = 0$ and let T_m be the mth time the process enters state 0 ($T_0 = 0$). Let $\tau_m = T_m - T_{m-1}$. For $j = 0,1,\ldots,k$ define

$$Y_m(j) = \sum_{n=T_{m-1}}^{T_m-1} f_j(X_m) \qquad (6)$$

We say that the process is in the mth cycle between times T_{m-1} and T_m-1 and that τ_m is the length of the mth cycle. Because the process regenerates itself at the times T_m (see [2]) we can conclude that $\{(Y_m(0),\ldots,Y_m(k), \tau_m), m \geq 1\}$ are i.i.d. (independent and identically distributed) random vectors. The importance of this in a simulation context is that the simulation run can then be broken up into randomly spaced i.i.d. blocks so that the techniques of classical statistics can be used to analyze the output (see [4]).

Suppose now that we simulate the process for M cycles. For each j define $\hat{r}_j(M)$ by

$$\hat{r}_j(M) = \sum_{m=1}^{M} Y_m(j) / \sum_{m=1}^{M} \tau_m . \qquad (7)$$

Define $\underset{\sim}{r}$ and $\hat{\underset{\sim}{r}}(M)$ to be the (k+1) dimensional column vectors with jth entries r_j and $\hat{r}_j(M)$ respectively. Let Σ be the symmetric matrix whose (i,j)th entry is

$$\sigma_{ij} = E[(Y_m(i) - r_i\tau_m)(Y_m(j) - r_j\tau_m)]. \qquad (8)$$

We assume that each element of Σ is finite and that Σ is positive definite. Let $N(0, \Sigma/E^2(\tau_1))$ denote a (k+1) dimensional random vector having the multivariate normal distribution with means 0 and a covariance matrix with elements $\sigma_{ij}/E^2(\tau_1)$. The following proposition is the key result for obtaining point estimates and confidence intervals for the r_j's.

(9) PROPOSITION. If $\pi|f_j| < \infty$ for all $j = 0,1,\ldots k$ and if $|\sigma_{ij}| < \infty$ for all i and $j = 0,1,\ldots,k$ then

$$r_j = E[Y_m(j)]/E[\tau_m] \qquad (10)$$

$$\hat{r}_j(M) \to r_j \quad \text{a.s. (almost surely)} \qquad (11)$$

$$\sqrt{M}(\hat{\underset{\sim}{r}}(M) - \underset{\sim}{r}) \Rightarrow N(\underset{\sim}{0}, \Sigma/E^2(\tau_1)). \qquad (12)$$

As a corollary to (12) if we let $\underset{\sim}{\beta} = (\beta(0),\ldots,\beta(k))$ and let

$$\sigma_k^2(\underset{\sim}{\beta}) = \underset{\sim}{\beta}\,\Sigma\,\underset{\sim}{\beta}' \qquad (13)$$

then the following central limit theorem is true;

$$\frac{\sqrt{M}(\underset{\sim}{\beta}\hat{\underset{\sim}{r}}(M) - \underset{\sim}{\beta}\underset{\sim}{r})}{\sigma_k(\underset{\sim}{\beta})/E(\tau_1)} \Rightarrow N(0,1) \qquad (14)$$

where N(0,1) denotes a normally distributed random variable with mean 0 and variance 1.

III. MULTIPLE ESTIMATES

We now turn to the first of the variance reduction techniques for Markov chains. Let us fix a function f and let $r = \pi f$. We are interested in obtaining short confidence intervals for r. To do this we form new functions f_j so that $r_j = \pi f_j = r$ for each j. As our first candidate for f_j let

$$f_j = P^j f \quad \text{for } j = 0,1,\ldots,k. \qquad (15)$$

We then have

$$\begin{aligned} r_j = \pi f_j &= \pi(P^j f) = \pi P(P^{j-1} f) \\ &= \pi(P^{j-1} f) \quad (\text{since } \pi = \pi P) \\ &= \pi f_{j-1} = r_{j-1} \end{aligned}$$

so that $r_j = r$ for all j. Define $\hat{r}_j(M)$ as in (7). By (11) $\hat{r}_j(M) \to r$ for each j. Now let $\underset{\sim}{\beta}$ be some vector so that

$$\sum_{j=0}^{k} \beta(j) = 1 \qquad (16)$$

and let $\hat{r}_\beta(M)$ be defined by

$$\hat{r}_\beta(M) = \sum_{j=0}^{k} \beta(j)\hat{r}_j(M). \qquad (17)$$

Then $\hat{r}_\beta(M) \to r$ a.s. as $M \to \infty$. Now using the central limit theorem in (14) and equation (16) we have

$$\frac{\sqrt{M}(\hat{r}_\beta(M)-r)}{\sigma_k(\underset{\sim}{\beta})/E(\tau_1)} \to N(0,1). \qquad (18)$$

We pick $\underset{\sim}{\beta} = \underset{\sim}{\beta}^*$ to minimize the variance term in (18) (this gives us the shortest confidence interval possible). If we let $\underset{\sim}{e}$ be a (k+1) dimensional row vector with each entry equal to 1 we then have;

$$\underset{\sim}{\beta}^* = \underset{\sim}{e}\,\underset{\sim}{\Sigma}^{-1}/\underset{\sim}{e}\,\underset{\sim}{\Sigma}^{-1}\underset{\sim}{e}', \qquad (19)$$

$$\sigma_k^2(\underset{\sim}{\beta}^*) = 1/\underset{\sim}{e}\,\underset{\sim}{\Sigma}^{-1}\underset{\sim}{e}', \qquad (20)$$

Let $R_k^2 = \sigma_k^2(\underset{\sim}{\beta}^*)/\sigma_{00}$. Combining $\hat{r}_0(M),\ldots,\hat{r}_k(M)$ in the above manner means that to obtain confidence intervals of equal length we need simulate only R_k^2 as many cycles than we would need if we used no variance reduction technique.

This method can be extended to continuous time Markov chains and semi-Markov processes by transforming them into discrete time Markov chains using the techniques of [7]. Table 1 lists the variance reductions for estimating the expected queue length, E(X), in a finite capacity M/M/1 queue. We let $\rho = \lambda/\mu$ where λ is the arrival rate and μ is the service rate. For ease of computation the capacity was chosen to be 14. It should be emphasized that these figures are actual calculations of variance reductions (based on the methods of [7]) and not simulation results.

TABLE 1

Variance Reductions for Finite Capacity M/M/1 Queue Using Multiple Estimates

ρ	R_1^2	R_2^2	R_3^2
.25	.1168	.0292	.0073
.50	.2341	.1121	.0524
.90	.6050	.2659	.1148
.95	.6880	.3425	.1607
.99	.7404	.4056	.2047

IV. MATRIX ITERATIVE TECHNIQUES

We now examine several other methods in an attempt to obtain greater variance reductions for a fixed amount of computation done before the simulation. The basic idea here is to partially solve a system of linear equations (with some matrix iterative technique) and then estimate the difference between the partial and true solutions via simulation.

We now assume that E is finite. Let $\underset{\sim}{y}$ be a vector with components $y(i)$ defined by

$$y(i) = E_i\Big[\sum_{n=0}^{T_1-1} f(X_n)\Big] \qquad (21)$$

where $E_i[\ ]$ denotes the expectation when $X_0 = i$. Let ${}_0P$ be a matrix defined by

$${}_0P_{ij} = \begin{cases} P_{ij} & \text{if } j \neq 0 \\ 0 & \text{if } j = 0 . \end{cases} \qquad (22)$$

It can then be shown (see [7]) that

$$\underset{\sim}{y} = \sum_{n=0}^{\infty} {}_0P^n f \qquad (23)$$

and that $\underset{\sim}{y}$ satisfies

$$\underset{\sim}{y} = f + {}_0P\,\underset{\sim}{y} . \qquad (24)$$

Recall that $r = y(0)/E(\tau_1)$. Let $\underset{\sim}{y}^j$ be our approximation to $\underset{\sim}{y}$ after j iterations of the Matrix iterative method. We then seek to find a function f_j such that

$$y(0) = y^j(0) + E_0\Big[\sum_{n=T_{m-1}}^{T_m-1} f_j(X_m)\Big] \qquad (25)$$

We can then set

$$Y_m(j) = y^j(0) + \sum_{n=T_{m-1}}^{T_m-1} f_j(X_m) \qquad (26)$$

and defining $\hat{r}_j(M)$ as in (7) we have $\hat{r}_j(M) \to r$ a.s. We can then proceed exactly as before to obtain variance reductions. Due to space limitations the derivations of $\underset{\sim}{y}^j$ and f_j for specific matrix iterative techniques are omitted here but may be found in [6]. Table 2 lists the calculated variance reductions for the finite capacity M/M/1 using the Gauss-Seidel iterative technique.

TABLE 2

Variance Reductions for Finite Capacity M/M/1 Queue Using Gauss-Seidel

ρ	R_1^2	R_2^2	R_3^2
.25	.0099	.0015	.0002
.50	.0720	.0408	.0207
.90	.5939	.3881	.1834
.95	.6789	.4794	.2414
.99	.7321	.5446	.2963

V. CONCLUSIONS

By viewing Tables 1 and 2 it can be seen that neither method dominates the other. The variance reductions will, in general, depend on the transition matrix P and the function f. The first method has the advantage that the variance reductions are independent of the return state whereas with the other methods care must be taken to pick a return state that yields good variance reductions. It is anticipated that frequently occurring states will produce the best variance reductions. Gauss-Seidel has the advantage that any $\underset{\sim}{y}^0$ may be used to initiate the iterative procedure. Generally speaking the closer $\underset{\sim}{y}^0$ is to $\underset{\sim}{y}$ the better the variance reduction will be. This suggests that one could simulate a small number of cycles to obtain an initial $\underset{\sim}{y}^0$ and then commence the Gauss-Seidel iterations.

In order for any of these methods to be computationally efficient the simulator must be able to compute (and store) the functions f_j. The amount of work involved in this could be considerable unless the transition matrix is sparse. It is for these types of processes that the methods are recommended. Further details on the methods, including more extensive numerical testing, may be found in [6].

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by Philip Heidelberger

Simulators frequently wish to estimate parameters of the limiting distribution of stable stochastic processes. Several new methods for reducing the variance of such estimates will be proposed and discussed. The methods are applicable to regenerative Markov processes in both discrete and continuous time as well as to semi-Markov processes. The methods are similar to the technique of multiple control variables yet differ in the important respect that it is not necessary to calculate the means of the controls. This is because the controls are chosen in such a way that their means actually equal the parameter of interest. The methods do require a certain amount of computation to be done before the simulation begins, although their cost should be relatively minor compared with that of the simulation. Numerical results demonstrating the effectiveness of the techniques for a simple queueing model are presented.